TENSOR PRODUCTS AND JOINT SPECTRA

BY

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ABSTRACT

We prove that the joint spectrum of the tensor product of several operators is the cartesian product of their spectra.

Let $X_1,\cdots,X_n$ be complex Banach spaces and let W be the completion of $X_1 \otimes \cdots \otimes X_n$ with respect to some crossnorm (cf., e.g., [1, 2]). Let I_k be the identity operator and A_k an arbitrary bounded operator on X_k, $1 \leqq k \leqq n$. Set $T_1 = A_1 \otimes I_2 \otimes \cdots \otimes I_n$, $T_2 = I_1 \otimes A_2 \otimes \cdots \otimes I_n$ and $T_k = I_1 \otimes \cdots \otimes I_{k-1} \otimes A_k \otimes \cdots \otimes I_n$ in general. The operators T_i obviously commute, and we have

$$\sigma(T_k) = \sigma(A_k), \ 1 \leqq k \leqq n. \tag{1}$$

Let B be the double commutant of $T_1,\cdots,T_n$, i.e., the set of those bounded operators on W which commute with any operator commuting with all of the T_k. B is a commutative Banach algebra. A complex vector $(\lambda_1,\cdots,\lambda_n)$ is in the joint resolvent $\rho\,(T_1,\cdots,T_n)$ of the T_k if there are operators C_k in B such that

$$\sum_1^n C_k(T_k - \lambda_k) = I = I_1 \otimes \cdots \otimes I_n. \tag{2}$$

Otherwise it is in the joint spectrum $\sigma(T_1,\cdots,T_n)$. In this note we shall give a proof of the following

THEOREM 1. *A complex vector $(\lambda_1,\cdots,\lambda_n)$ is in $\sigma\,(T_1,\cdots,T_n)$ if and only if $\lambda_k \in \sigma(A_k)$, $1 \leqq k \leqq n$. In symbols*

$$\sigma(T_1,\cdots,T_n) = \prod_1^n \sigma(A_k). \tag{3}$$

Let $U(z_1,\cdots,z_n)$ be a rational function without singularities on $\sigma(T_1,\cdots T_n)$ It was shown in [3] that

$$\sigma[U(T_1,\cdots,T_n)] = U[\sigma(T_1,\cdots,T_n)]. \tag{4}$$

An immediate consequence of (4) and Theorem 1 is

Received December 17, 1969.

COROLLARY 1. *A complex number μ is in $\sigma[U(T_1,\cdots,T_n)]$ if and only if there are complex numbers $\lambda_1,\cdots,\lambda_n$ such that $\lambda_k \in \sigma(A_k)$, $1 \leqq k \leqq n$, and $\mu = U(\lambda_1,\cdots,\lambda_n)$.*

Corollary 1 was proved in [4] without the use of Theorem 1. The case $U(z_1,z_2) = z_1 z_2$ was previously considered in [5]. It might appear that the approach of the present paper to proving Corollary 1 is simpler than that of [4]. This is illusory since our present approach needs the Gelfand representation theorem (in the proof of (4)) while that of [4] does not.

We now give the

Proof of Theorem 1. If $\lambda_k \in \rho(A_k)$ for some k, then $(\lambda_1,\cdots,\lambda_n)$ is in $\rho(T_1,\cdots,T_n)$. For we can take $C_k = (T_k - \lambda_k)^{-1}$ and $C_j = 0$ for $j \neq k$ in (2). Thus

$$\sigma(T_1,\cdots,T_n) \subset \prod_1^n \sigma(A_k). \tag{5}$$

Thus it suffices to show that $\lambda_k \in \sigma(A_k)$ for each k imples that $(\lambda_1,\cdots,\lambda_n)$ is in $\sigma(T_1,\cdots,T_n)$. We may assume each $\lambda_k = 0$. Since $0 \in \sigma(A_k)$, we have either a sequence $\{\chi_{km}\}$ of elements in X_k such that

$$\| \chi_{km} \| = 1, A_k\chi_{km} \to 0 \text{ as } m \to \infty \tag{6}$$

or a sequence $\{\chi'_{km}\}$ of elements in X'_k such that

$$\| \chi'_{km} \| = 1,\ A'_k\chi'_{km} \to 0 \text{ as } m \to \infty \tag{7}$$

(here X'_k denotes the dual space of X_k and A_k' the conjugate of A_k), or both (6) and (7) hold. By reordering the spaces X_k if necessary, we may assume that there is an integer t such that $0 \leqq t \leqq n$ and (6) holds for $1 \leqq k \leqq t$ while (7) holds for $t < k \leqq n$.

For $1 \leqq k \leqq t$ let $\chi'_{km} \in X'_k$ be such that

$$\| \chi'_{km} \| = 1 \text{ and } \chi'_{km}(\chi_{km}) = 1, \tag{8}$$

and for $t < k \leqq n$ let $\chi_{km} \in X_k$ be such that

$$\| \chi_{km} \| = 1 \text{ and } |\chi'_{km}(\chi_{km})| > 1 - \frac{1}{m}. \tag{9}$$

Set

$$\chi_m = \chi_{1m} \otimes \cdots \otimes \chi_{1m},\ \chi'_m = \chi'_{1m} \otimes \cdots \otimes \chi'_{nm}.$$

Then by (8) and (9)

$$|\chi'_m(\chi_m)| \geqq \left(1 - \frac{1}{m}\right)^n \to 1 \text{ as } m \to \infty. \tag{10}$$

Now suppose there existed operators $C_k \in W$ such that

$$\sum_1^n C_k T_k = I. \tag{11}$$

Then

$$\chi_m'(\sum_k C_k T_k \chi_m) = \sum_{k=1}^{t} \chi_m'(C_k T_k \chi_m) + \sum_{k=t+1}^{n} T_k' \chi_m'(C_k \chi_m).$$

Thus by (11)

$$\left| \chi_m'(\chi_m) \right| \leqq \sum_{k=1}^{t} \|C_k\| \, \|A_k \chi_{km}\| + \sum_{k=t+1}^{n} \|C_k\| \, \|A_k' \chi_{km}'\|$$

$$\to 0 \text{ as } m \to \infty.$$

This contradicts (10). Thus (11) cannot hold and the proof is complete.

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